A parabolic PDE

The heat equation on a long, thin rod is

$$\frac{\partial u}{\partial t}(x,t) = \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t)$$

with boundary conditions

$$u(0,t) = 0$$

$$u(l,t) = 0$$

$$u(x,0) = f(x)$$

We solve this equation by the method of finite differences by replacing the derivative terms with difference quotients on a set of grid points (x_i , t_j):

$$\frac{\partial u}{\partial t}(x_i,t_j) = \frac{u(x_i,t_j + k) - u(x_i,t_j)}{k} + O(k)$$

$$\frac{\partial^2 u}{\partial x^2}(x_i,t_j) = \frac{u(x_i + h,t_j) - 2\, u(x_i,t_j) + u(x_i - h,t_j)}{h^2} + O(h^2)$$

Here the grid points are determined by the formulas

$$x_i = i\, h$$

$$t_j = j\, k$$

$$h = \frac{l}{m}$$

where i ranges from 0 to m.

The forward difference method

Plugging these estimates into the PDE and using the notation $w_{i,j}$ for our estimates for $u(x_i,t_j)$ we get that the $w_{i,j}$ satisfy a system of equations

$$\frac{w_{i,j+1} - w_{i,j}}{k} - \alpha^2 \frac{w_{i+1,j} - 2\, w_{i,j} + w_{i-1,j}}{h^2} = 0$$

Solving these equations for $w_{i,j+1}$ gives

$$w_{i,j+1} = \left(1 - \frac{2\,\alpha^2\,k}{h^2}\right) w_{i,j} + \alpha^2\,\frac{k}{h^2}\,\left(w_{i-1,j} + w_{i+1,j}\right) \qquad (1)$$

This can also be written as a simple matrix equation

$$\mathbf{w}^{(j+1)} = A\ \mathbf{w}^{(j)}$$

The initial conditions give us that

$$w_{0,j} = 0$$

$$w_{m,j} = 0$$

$$w_i,0 = f(x_i)$$

Since all of the terms on the right hand side of equation (1) are known for $j = 0$, we can compute $w_{i,1}$ for all i. Similarly, we can compute all of the $w_{i,j}$ terms we want by just iterating over j and i.

In practice, though, this simple method does not work well, since the solution generated by this method not numerically stable: small errors in the initial function $f(x)$ can translate into large errors in our estimate for $u(x,t)$ for large t.

The backward difference method

An alternative approach is to use the difference quotient

$$\frac{\partial u}{\partial t}(x_i,t_j) = \frac{u(x_i,t_j) - u(x_i,t_j - k)}{k} + O(k)$$

in the original equation. This changes our system of equations to

$$\frac{w_{i,j} - w_{i,j-1}}{k} - \alpha^2\,\frac{w_{i+1,j} - 2\,w_{i,j} + w_{i-1,j}}{h^2} = 0$$

or

$$-\,\lambda\,w_{i-1,j} + \left(1 + 2\,\lambda\right) w_{i,j} - \lambda\,w_{i+1},j = w_{i,j-1}$$

where $\lambda = \alpha^2\,k/h^2$.

Now to solve for $w_{i,j}$ for a given j and $1 \le i \le m$-1 we have to solve a system of m-1 equations in m-1 unknowns.

A hyperbolic PDE

The wave equation for a vibrating string is

$$\frac{\partial^2 u}{\partial t^2}(x,t) - \alpha^2 \frac{\partial^2 u}{\partial x^2}(x,t) = 0$$

The boundary conditions for this problem specify that the string is fixed at both ends of the interval $0 \leq x \leq l$, and also specify the initial displacement and velocity of the string at time $t = 0$:

$$u(0,t) = u(l,t) = 0$$

$$u(x,0) = f(x)$$

$$\frac{\partial u}{\partial t}(x,0) = g(x)$$

We solve this equation by the method of finite differences by replacing the derivative terms with difference quotients on a set of grid points $(x_i\ ,\ t_j)$:

$$\frac{\partial^2 u}{\partial t^2}(x_i,t_j) = \frac{u(x_i,t_j + k) - 2\ u(x_i,t_j) + u(x_i,t_j - k)}{k^2} + O(k^2)$$

$$\frac{\partial^2 u}{\partial x^2}(x_i,t_j) = \frac{u(x_i + h,t_j) - 2\ u(x_i,t_j) + u(x_i - h,t_j)}{h^2} + O(h^2)$$

Here the grid points are determined by the formulas

$$x_i = i\ h$$

$$t_j = j\ k$$

$$h = \frac{l}{m}$$

where i ranges from 0 to m.

Plugging these estimates into the PDE and using the notation $w_{i,j}$ for our estimates for $u(x_i,t_j)$ we get that the $w_{i,j}$ satisfy a system of equations

$$\frac{w_{i,j+1} - 2\ w_{i,j} + w_{i,j-1}}{k^2} - \alpha^2 \frac{w_{i+1,j} - 2\ w_{i,j} + w_{i-1,j}}{h^2} = 0$$

If we solve this equation for $w_{i,j+1}$ we get

$$w_{i,j+1} = 2\,(1 - \lambda^2)\,w_{i,j} + \lambda^2\,(w_{i+1,j} + w_{i-1,j}) - w_{i,j-1}$$

This update rule allows us to compute an approximation for $u(x_i,t_{j+1})$ in terms of estimates at t_j and t_{j-1}. The only problem with this scheme is the case $j = 0$. To compute estimates for $u(x_i,t_1)$ we would need values for the solution at $t_0 = 0$ and $t_{-1} = -k$. The boundary condition $u(x,0) = f(x)$ gives us the first set of values, but we don't have values to tell us what u is doing at $t = -k$.

One fix for this problem is to start with a power series expansion for $u(x,t)$ in t about t = 0:

$$u(x_i,t_1) = u(x_i,0) + k\,\frac{\partial u}{\partial t}(x_i,0) + \frac{k^2}{2}\,\frac{\partial^2 u}{\partial t^2}(x_i,0) + O(k^3)$$

The first derivative term is given by one of the initial conditions. We can handle the second derivative term by solving the differential equation for $\frac{\partial^2 u}{\partial t^2}(x_i,0)$:

$$\frac{\partial^2 u}{\partial t^2}(x_i,0) = \alpha^2\,\frac{\partial^2 u}{\partial x^2}(x_i,0) = \alpha^2\,\frac{\mathrm{d}^2 f}{\mathrm{d}x^2}(x)$$

Putting this all together gives us

$$u(x_i,t_1) = u(x_i,0) + k\,g(x_i) + \frac{\alpha^2\,k^2}{2}\,f''(x_i) + O(k^3)$$

Thus we have

$$w_{i,0} = f(x_i)$$

$$w_{i,1} = w_{i,0} + k\,g(x_i) + \frac{\alpha^2\,k^2}{2}\,f''(x_i)$$

$$w_{i,j+1} = 2\,(1 - \lambda^2)\,w_{i,j} + \lambda^2\,(w_{i+1,j} + w_{i-1,j}) - w_{i,j-1}$$