Boundary Value Problems

A second order boundary value problem on a closed interval $a \leq x \leq b$ is a differential equation that takes the form

$$y'' = f(x,y,y')$$

$$y(a) = \alpha$$

$$y(b) = \beta$$

Given the similarity between this problem and the second order initial value problem one would think that there is not much new here. To a certain extent this is true: as we will see below we can use techniques for solving initial value problems to attack this problem. However, these techniques are not simple.

The linear case

An important special case of the problem we are studying here is the second order linear boundary value problem. In this version the function f takes a special form, which makes it easier to deal with the problem of matching the boundary conditions.

$$y'' = p(x)\, y' + q(x)\, y + r(x)$$

What makes this form easy to work with is the fact that the right hand side is linear in the function $y(x)$. Suppose that $y_1(x)$ is a solution of

$$y'' = p(x)\, y' + q(x)\, y + r(x)$$

and $y_2(x)$ is a solution of

$$y'' = p(x)\, y' + q(x)\, y$$

Any linear combination

$$y(x) = c_1\, y_1(x) + c_2\, y_2(x)$$

is also a solution of the original equation:

$$y'' = c_1\, y_1'' + c_2\, y_2''$$

$$= c_1 \left(p(x)\, y_1' + q(x)\, y_1 + r(x)\right) + c_2 \left(p(x)\, y_2' + q(x)\, y_2\right)$$

$$= p(x) \left(c_1\, y_1(x) + c_2\, y_2(x)\right)' + q(x) \left(c_1\, y_1(x) + c_2\, y_2(x)\right) + r(x)$$

$$= p(x)\, y' + q(x)\, y + r(x)$$

We can take advantage of this property of the linear problem to solve the boundary value problem. Our approach is to find a function $y_1(x)$ that solves the initial value problem

$$y'' = p(x)\, y' + q(x)\, y + r(x)$$

$$y(a) = \alpha$$

$$y'(a) = 0$$

This is a conventional initial value problem, and can be solved by any of the methods from chapter 5.

We then solve a second initial value problem. $y_2(x)$ is the solution to

$$y'' = p(x)\, y' + q(x)\, y$$

$$y(a) = 0$$

$$y'(a) = 1$$

From what we saw above,

$$y(x) = c_1\, y_1(x) + c_2\, y_2(x)$$

Solves the differential equation. Note also that

$$y(a) = c_1\, \alpha + c_2\, 0 = c_1\, \alpha$$

This tells us that we must pick $c_1 = 1$. To match the boundary condition at the point $x = b$ we require that

$$y(b) = y_1(b) + c_2\, y_2(b) = \beta$$

This is equivalent to requiring that

$$c_2 = \frac{\beta - y_1(b)}{y_2(b)}$$

Thus,

$$y(x) = y_1(x) + \frac{\beta - y_1(b)}{y_2(b)}\, y_2(x)$$

solves the differential equation and matches the boundary conditions.

The nonlinear case

The technique we saw above works only in the case of a linear equation, because it makes essential use of the fact that a linear combination of solutions is also a solution. For the more general, nonlinear $f(x,y,y')$ this property no longer holds.

Even so, we will want to develop a method that uses the solution of an initial value problem to solve the boundary value problem.

Here is a simple idea that will do this. Let $y(x)$ be the solution to the initial value problem

$$y'' = f(x,y,y')$$

$$y(a) = \alpha$$

$$y'(a) = t$$

If we can find a value of t that causes $y(b) = \beta$ we will have solved our problem.

To emphasize the dependence of $y(x)$ on our choice of t, we will write $y(x,t)$.

The problem we have to solve now is finding a value of t such that

$$y(b,t) = \beta$$

This idea is called the *shooting method*, because we are shooting a function with a particular slope at $x = a$ and trying to hit a target value at $x = b$. The shooting method boils down to a root-finding problem. We seek a value of t such that

$$v(t) = y(b,t) - \beta$$

has a root. The only complication here is that $v(t)$ is a very unwieldy function to work with. Consider what is required to evaluate $v(t)$ for some choice of t:

1. Specify your choice for t.
2. Use a numerical solution technique to solve the initial value problem

$$y'' = f(x,y,y')$$

$$y(a) = \alpha$$

$$y'(a) = t$$

3. Evaluate that numerical solution at b to get $v(t) = y(b,t) - \beta$.

This makes for a very expensive and cumbersome function evaluation!

Given the practical difficulty of evaluating $v(t)$ we will have to use a root finding technique that is easy to use. One possibility is to use the secant method. In this method we pick two values of t, t_1 and t_2, and construct the secant line passing through the points $(t_1,v(t_1))$ and $(t_2,v(t_2))$ and then determine the t_3 where that secant line crosses the axis. We then repeat the process with t_2 and t_3 to get a t_4, and so on. This will produce a sequence of approximate values for t that should converge to the root.

Using Newton's method

Can we use Newton's method to find a root for $v(t)$? At first glance this seems impossible, because the function $v(t)$ is a ridiculously complicated function. How will we ever compute $v'(t)$ to apply the Newton iteration formula?

$$t_{k+1} = t_k - \frac{v(t_k)}{v'(t_k)}$$

Amazingly, it is actually possible to compute $v'(t_k)$. The trick is to notice that $v(t)$ comes from the solution of a differential equation, and that we can differentiate that equation with respect to the parameter t:

$$\frac{\partial y''(x,t)}{\partial t} = \frac{\partial f(x,y(x,t),y'(x,t))}{\partial t} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y(x,t)}{\partial t} + \frac{\partial f}{\partial y'}\frac{\partial y'(x,t)}{\partial t}$$

Two facts help us to rewrite this. The first is that the variables x and t are independent of each other, so that

$$\frac{\partial x}{\partial t} = 0$$

The second fact is that

$$\frac{\partial y'(x,t)}{\partial t} = \frac{\partial}{\partial t}\frac{\partial}{\partial x} y(x,t) = \frac{\partial}{\partial x}\frac{\partial y(x,t)}{\partial t}$$

and also

$$\frac{\partial y''(x,t)}{\partial t} = \frac{\partial^2}{\partial x^2}\frac{\partial y(x,t)}{\partial t}$$

These facts transform the equation above into

$$\frac{\partial^2}{\partial x^2}\frac{\partial y(x,t)}{\partial t} = \frac{\partial f}{\partial y}\frac{\partial y(x,t)}{\partial t} + \frac{\partial f}{\partial y'}\left(\frac{\partial}{\partial x}\frac{\partial y(x,t)}{\partial t}\right)$$

If we introduce

$$z(x,t) = \frac{\partial y(x,t)}{\partial t}$$

This becomes an equation

$$z''(x,t) = \frac{\partial f}{\partial y} z(x,t) + \frac{\partial f}{\partial y'} z'(x,t)$$

The original initial conditions

$$y(a) = \alpha$$

$$y'(a) = t$$

become

$$z(a,t) = 0$$

$$z'(a,t) = 1$$

What we have now is a coupled system of two equations in two unknowns, $y(x,t)$ and $z(x,t)$:

$$y''(x,t) = f(t,y(x,t),y'(x,t))$$

$$y(a,t) = \alpha$$

$$y'(a,t) = t$$

$$z''(x,t) = \frac{\partial f}{\partial y} z(x,t) + \frac{\partial f}{\partial y'} z'(x,t)$$

$$z(a,t) = 0$$

$$z'(a,t) = 1$$

The equations are coupled, because y and y' terms will still appear in the equation for z. This mess can be handled by techniques from chapter 5. Specifically, we can convert this into a first order system of four equations in four unknowns. Solving this system will allow us to compute value for $y(b,t)$ and $z(b,t)$.

Finally, note that

$$v'(t) = \frac{\partial}{\partial t}\left(y(b,t) - \beta\right) = z(b,t)$$

We now have everything we need to apply Newton's method to the original problem of finding a root of $v(t)$.

1. Given a t_k, we start by solving the system

$$y''(x,t_k) = f(x,y(x,t_k),y'(x,t_k))$$

$$z''(x,t_k) = \frac{\partial f}{\partial y} z(x,t_k) + \frac{\partial f}{\partial y'} z'(x,t_k)$$

$$y(a,t_k) = \alpha$$

$$y'(a,t_k) = t_k$$

$$z(a,t_k) = 0$$

$$z'(a,t_k) = 1$$

2. We use our numerical solution technique to estimate $y(b,t_k)$ and $z(b,t_k)$.

3. If $|y(b,t_k) - \beta|$ is small enough, we stop.

4. Otherwise, we compute the next t and repeat:

$$t_{k+1} = t_k - \frac{v(t_k)}{v'(t_k)} = t_k - \frac{y(b,t_k) - \beta}{z(b,t_k)}$$